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Necessary and Sufficient Conditions for Determining a Hill's Equation from Its Spectrum*

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A Hill's equation is an equation of the form

$$y'' + [\lambda - q(z)]y = 0. \tag{1}$$

We assume $q(z + \pi) = q(z)$, where $q(z)$ is integrable over $[0, \pi]$. Without loss of generality it is customary to assume that $\int_0^\pi q(z)\,dz = 0$. The discriminant of (1) is defined by

$$\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$$

where y_1 and y_2 are solutions of (1) satisfying

$$y_1(0) = y_2'(0) = 1$$

and

$$y_1'(0) = y_2(0) = 0.$$

Pertinent information about the analytic structure of the discriminant can be found in Magnus and Winkler [1]. $\Delta(\lambda)$ is an entire function of order 1/2 and $\Delta(\lambda) - 2$ has infinitely many zeros with no finite limit point. To each zero there corresponds a solution of (1) satisfying

$$y(\pi) = y(0) \qquad \text{and} \qquad y'(\pi) = y'(0).$$

These conditions, combined with (1), define a self-adjoint boundary value problem. It has only real eigenvalues which are the zeros of $\Delta(\lambda) - 2$. They are denoted by λ_i ($i = 0, 1, 2,...$) and are arranged so that

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots.$$

Similarly, the eigenvalues corresponding to the boundary condition

$$y(\pi) = -y(0) \qquad \text{and} \qquad y'(\pi) = -y'(0)$$

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are the zeros of $\Delta(\lambda) + 2$ which are denoted by λ_i' $(i = 1, 2, 3,...)$ and are arranged so that

$$\lambda_1' \leqslant \lambda_2' < \lambda_3' \leqslant \lambda_4' < \cdots,$$

the two sequences are interlaced so that

$$\lambda_0 < \lambda_1' \leqslant \lambda_2' < \lambda_1 \leqslant \lambda_2 < \lambda_3' \leqslant \lambda_4' < \cdots,$$

the following intervals are now formed:

$$(-\infty, \lambda_0], (\lambda_0, \lambda_1'), [\lambda_1', \lambda_2'], (\lambda_2', \lambda_1), [\lambda_1, \lambda_2],....$$

In the intervals of type $(-\infty, \lambda_0]$ and $[\lambda_{2n-1}, \lambda_{2n}]$ we have $\Delta \geqslant 2$. In those of type $(\lambda_{2n}, \lambda'_{2n+1})$ and $(\lambda'_{2n}, \lambda_{2n-1})$ we have $|\Delta| < 2$. In intervals of type $[\lambda'_{2n-1}, \lambda'_{2n}]$ we have $\Delta \leqslant -2$.

For values of λ such that $|\Delta| > 2$, (1) has no solution which is bounded for all real z. When $|\Delta| = 2$, there exists at least one bounded solution. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z.

Therefore, the intervals for which $|\Delta| < 2$ are called stability intervals, while the remaining intervals are called instability interval. All instability intervals are finite except for $(-\infty, \lambda_0]$.

In [2] it was proved that if $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form

$$q^{(2n)} + H(q, q',..., q^{(2n-2)}) = 0 \tag{2}$$

where H is a polynomial of maximal degree $n + 2$.

Borg [3], Hochstadt [4], and Ungar [5] proved this theorem for the case $n = 0$, i.e., when all finite instability intervals vanish, and found that

$$q(z) = 0. \tag{3}$$

For the case $n = 1$, Hochstadt [6] showed that $q(z)$ is an elliptic function which satisfies

$$q'' = 3q^2 + Aq + B \tag{4}$$

where A and B are constants.

For the case $n = 2$, Goldberg [2] showed that the explicit expression of (2) is

$$q^{(4)} = 10qq'' + Aq'' + 5(q')^2 - 10q^3 + Bq^2 + Cq + D, \tag{5}$$

where A, B, C and D are constants.

Erdelyi [7] investigated a Hill's equation where $q(z)$ is a Lamé function and discovered situations where all but a finite number of finite instability intervals vanish. (4) provides a converse to some of his results.

For an infinite class of Korteweg–de Vries equations of the form

$$q_t = K_n(q, q_z, \ldots, \partial^{2n+1}q/\partial z^{2n+1}), \qquad (n = 0, 1, 2, \ldots),$$

Lax [8] has found that a function q satisfying

$$K_n(q, \ldots, \partial^{2n+1}q/\partial z^{2n+1}) = 0, \tag{6}$$

requires (1) to have no more than n finite instability intervals. Equations (3), (4) and (5) are equivalent to (6) when n is 0, 1 and 2 respectively and hence provide a converse to Lax's result for these cases.

In this paper, we extend the above results to show that (2) and (6) are equivalent for all values of n. Hence we have necessary and sufficient conditions which the periodic potential function $q(z)$ must satisfy when n finite instability intervals of (1) fail to vanish.

Formally, this result is stated as:

THEOREM. *Let $q(z)$ be real, integrable and periodic. Then precisely n finite instability intervals of* (1) *fail to vanish if and only if $q(z)$ satisfies the* nth *order Korteweg–de Vries equation* (6).

Hochstadt [6] also proved that $q(z)$ is infinitely differentiable when n finite instability intervals fail to vanish. This result is assumed throughout this paper.

Lenard [9] derives the class of Korteweg–de Vries equations by using the following scheme: Let

$$u_t = P(u, u_x, \ldots)$$

where u is the potential in the Schrödinger equation

$$\psi_{xx} - (u - \lambda)\psi = 0. \tag{8}$$

Assuming

$$P\psi^2 = (A\psi_x^2 + B\psi\psi_x + C\psi^2)_x$$

where A, B and C are functions of u and λ, and using (7), he obtains

$$A_x + B = 0$$

$$2A(u - \lambda) + B_x + 2C = 0$$

and

$$P = B(u - \lambda) + C_x .$$

Eliminating B and C gives

$$P = (1/2)\, A_{xxx} - 2(u - \lambda)\, A_x - u_x A.$$

By assuming A has the form

$$A = \lambda^n \left(A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + \cdots + \frac{A_n}{\lambda^n}\right) = \sum_{i=0}^{n} A_i \lambda^{n-i},$$

and requiring A and P to be independent of λ, it follows that

$$P = \tfrac{1}{2}A_n''' - 2uA_n' - u'A_n \tag{9}$$

and

$$A'_{m+1} = -\tfrac{1}{4}A_m''' + uA_m' + \tfrac{1}{2}u'A_m, \tag{10}$$

(where $'$ denotes differentiation with respect to x). From (9) and (10) it is clear that

$$P = -2A'_{m+1} \qquad (m = 0, 1, \ldots, n).$$

Hence we have infinitely many expressions for (7) which are generated by recursion formula (10), taking $A_0 = 1$.

We now proceed to show that (2) is equivalent to

$$A'_{n+1} = 0$$

for all values of n.

Let $u_2(t)$ be the solution of

$$u'' + [\lambda - q(z + \tau)]u = 0$$

which satisfies $u_2(0) = 0$ and $u_2'(0) = 1$.

Based on Hochstadt's result [6] that q is necessarily infinitely differentiable when n finite instability intervals fail to vanish, Goldberg [2] showed that $u_2(\pi)$ can be written in the following convergent series

$$u_2(\pi) = \frac{\sin \lambda^{1/2}\pi}{\lambda^{1/2}} \sum_{n=0}^{\infty} \frac{S_n(\tau)}{\lambda^n} + \cos \lambda^{1/2}\pi \sum_{n=1}^{\infty} \frac{R_n}{\lambda^n}. \tag{11}$$

Here R_n is a constant independent of τ and S_n is of the form

$$S_n = \frac{(-1)^{n-1}}{2^{2n}} q^{(2n-2)} + S_n{}^*$$

where $S_n{}^*$ is a polynomial of maximal degree

$$n + 1 \quad \text{in} \quad q, q', \ldots, q^{(2n-4)}.$$

He further showed that (2) is derived from

$$\sum_{j=0}^{n} [\sigma_j(0)\, S_{n+1-j}(\tau) - S_{n+1-j}(0)\, \sigma_j(\tau)] = 0,$$

where $\sigma_0(\tau) = 1$ and $\sigma_k(\tau)$ is a linear combination of $S_0(\tau)$, $S_1(\tau),\ldots, S_k(\tau)$ for $k \geqslant 1$. This is rewritten as

$$\sum_{i=0}^{n+1} C_i S_i(\tau) = 0, \tag{12}$$

where C_i ($i = 0, 1,\ldots, n+1$) is a constant. Noting that

$$u_2(t) = y_1(\tau)\, y_2(t+\tau) - y_2(\tau)\, y_1(t+\tau),$$

we differentiate $u_2(\pi)$ three times with respect to τ and obtain

$$(\partial^3/\partial\tau^3)\, u_2(\pi) = 2q'(\tau)\, u_2(\pi) + 4\,[q(\tau) - \lambda]\,(\partial/\partial\tau)\, u_2(\pi). \tag{13}$$

Upon substituting (11) into (13) we obtain

$$\sum_{n=0}^{\infty} \frac{\sin \lambda^{1/2}\pi}{\lambda^{1/2}} \left[\frac{S_n'''}{2\lambda^n} - \frac{2qS_n{}'}{\lambda^n} + \frac{2\lambda S_n{}'}{\lambda^n} - \frac{q'S_n}{\lambda^n}\right] = 0,$$

which implies that

$$S'_{n+1} = -\tfrac{1}{4}S_n''' + qS_n{}' + q'S_n/2 \tag{14}$$

for all values of n.

Equations (14) and (10) are the same relation and hence S_n is generated by the same formula as A_n. By including an arbitrary constant of integration we obtain for $n = 0$, 1, 2 and 3.

$$\begin{aligned}
A_0 &= 1\\
A_1 &= \tfrac{1}{2}q + C_1\\
A_2 &= \tfrac{1}{8}[3q^2 - q''] + C_1[\tfrac{1}{2}q] + C_2\\
A_3 &= \tfrac{1}{32}[q^{(4)} - 5(q')^2 - 10qq'' + 10q^3] + C_1\,\tfrac{1}{8}[3q^2 - q''] + C_2[\tfrac{1}{2}q] + C_3\,.
\end{aligned} \tag{15}$$

In generating S_n we take $C_i = 0$ ($i = 1, 2,\ldots$). This choice is justified by comparing $S_n(0)$ with the series representation of $y_2(\pi)$ which includes no constants independent of q.

From (15) and the above comment, it is clear that

$$A_n(\tau) = \sum_{i=0}^{n} C_i S_i(\tau).$$

Hence (12) can be expressed as

$$A_{n+1} = 0.$$

By differentiating this expression we obtain $A'_{n+1} = 0$ which is precisely Eq. (6).

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